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Path-integral treatment of ring-shaped topological defects

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Abstract. Explicit path integration is carried out in a space with a ring-shaped topological defect using toroidal coordinates. The toroidal Aharonov–Bohm experiment is taken as an example.

1. Introduction

The formulation of quantum mechanics in spaces with topological defects such as a punctured plane or a ring in three-dimensional space can have interesting applications in statistical mechanics for polymers [1], condensed matter physics and elementary particles [2]. Certainly, Feynman's path integral formalism [3] is a most convenient tool to use to treat this class of problems. Being a sum over all possible histories of the particle, it is naturally sensitive to such topological defects.

An example which has received much attention in the literature [4–7] is the punctured plane where the hole, or singularity, makes the physical space multiply connected. To deal with this, the propagator is expressed as a path integral in the covering space which is simply connected [5]. The path integral in this case becomes a sum of partial propagators $K_n(\boldsymbol{r}'', \boldsymbol{r}'; \tau)$, each corresponding to homotopically inequivalent paths winding n times around the singularity, i.e.

$$K(\boldsymbol{r}'', \boldsymbol{r}'; \tau) = \sum_{n=-\infty}^{\infty} K_n(\boldsymbol{r}'', \boldsymbol{r}'; \tau). \tag{1.1}$$

This method has been applied to treat the Aharonov–Bohm (AB) [8] effect due to a solenoidal flux since the impenetrable solenoid in three-dimensional space, when symmetrically sectioned, leads to a singularity in a plane.

Another example of interest is the torus-shaped topological defect (figure 1). Once again, the space is multiply connected, and trajectories going from the source S to a detector D, but with different windings around the torus, can no longer be deformed into others. Such paths belong to homotopically inequivalent classes. It is for this interesting case that a path-integral treatment is presented in this paper. In view of the geometry of the system, we use the toroidal coordinate system and find that the path integral can be evaluated by applying the technique of path-dependent time transformation [9] together with results of path integration over the SU(1,1) group manifold [10].

As an application, we consider the toroidal AB [11] effect. In contrast to the solenoidal AB effect, which has been plagued with flux leakage near the ends of the solenoid, the

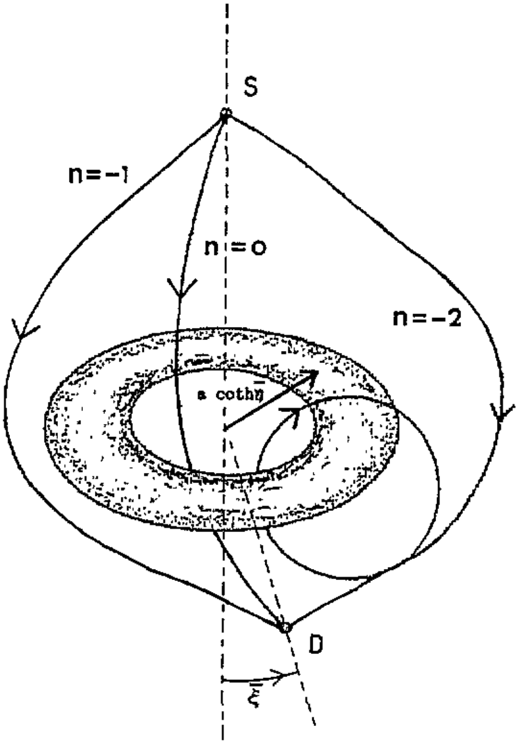

Figure 1. Paths with different winding number n are homotopically inequivalent. The limit, $\bar{\eta} \to \infty$, corresponds to an infinitesimally thin ring of radius a.

toroidal case gives a finite and closed geometry that effectively confines the flux. This was in fact applied by Tonomura *et al* [11] in the verification of the AB effect where a magnet of toroidal geometry, coated with a superconductor, was used to prevent electron penetration and flux leakage. To our knowledge, there has been no explicit formulation of the toroidal AB effect in the path-integral formalism, and it is certainly worthwhile to look into this problem where the implications of (1.1) can further be explored.

In the next section, we briefly review the toroidal coordinate system and express the free particle propagator in toroidal coordinates. In section 3, the case with a ring-shaped defect is considered and the path integral is evaluated. In section 4, the energy Green function, which is the Fourier transform of the propagator, is evaluated in closed form. Section 5 discusses the interference pattern in the toroidal AB experiment where magnetic flux is confined in a ring. A summary of results and possible extensions are given in section 6.

2. The propagator in toroidal coordinates

In the toroidal coordinate system, a point is specified by the intersection of surfaces given by [12]:

(i) $\eta = \eta_0$ $(0 \leqslant \eta < \infty)$, which is a torus with the axial circle in the x–y plane and centred at the origin, of radius $a \coth \eta_0$ and circular cross section of radius $a \operatorname{cosech} \eta_0$. The limit $\eta \to \infty$ corresponds to an infinitesimally thin ring of radius a, and $\eta \to 0$ corresponds to the z-axis.

(ii) $\xi = \xi_0$ $(0 \leqslant \xi < 2\pi)$, which is that section of a spherical surface of radius $a \operatorname{cosec} \xi_0$, centred at $x = y = 0$, $z = a \cot \xi_0$, that is above the x–y plane for the range $0 \leqslant \xi < \pi$; the rest of the same sphere and below the x–y plane for the range $\pi < \xi < 2\pi$. The part of the x–y plane outside the circle $r = a$, $z = 0$ corresponds to $\xi = 0$ or 2π; the rest of the x–y plane inside the circle corresponds to $\xi = \pi$.

(iii) $\phi = \phi_0$ $(0 \leqslant \phi < 2\pi)$, which is the usual azimuthal plane in the cylindrical coordinate system.

From Cartesian coordinates, the transformation equations are

$$x = \frac{a \sinh \eta \cos \phi}{\cosh \eta - \cos \xi} \qquad y = \frac{a \sinh \eta \sin \phi}{\cosh \eta - \cos \xi} \qquad z = \frac{a \sin \xi}{\cosh \eta - \cos \xi}. \tag{2.1}$$

The distance between two points in toroidal coordinates is given by

$$\mathrm{d}s^2 = (h_\eta \mathrm{d}\eta)^2 + (h_\xi \, \mathrm{d}\xi)^2 + (h_\phi \, \mathrm{d}\phi)^2 \tag{2.2}$$

where

$$h_\eta = \frac{a}{\cosh \eta - \cos \xi} \qquad h_\xi = \frac{a}{\cosh \eta - \cos \xi} \qquad h_\phi = \frac{a \sinh \eta}{\cosh \eta - \cos \xi} \tag{2.3}$$

and the volume element is

$$\mathrm{d}^3 \boldsymbol{r} = h_\eta h_\xi h_\phi \, \mathrm{d}\eta \, \mathrm{d}\xi \, \mathrm{d}\phi = \frac{a^3 \sinh \eta \, \mathrm{d}\eta \, \mathrm{d}\xi \, \mathrm{d}\phi}{(\cosh \eta - \cos \xi)^3}. \tag{2.4}$$

To set up the propagator as a path integral in toroidal coordinates, we first consider a free particle of mass μ. In Feynman's prescription, the propagator is given as the sum over all possible histories of the particle:

$$K(\boldsymbol{r}'', \boldsymbol{r}', \tau) = \lim_{N \to \infty} \int \prod_{j=1}^{N} \exp\left(\frac{\mathrm{i}}{\hbar} S_j\right) \left(\frac{\mu}{2\pi \mathrm{i} \hbar \tau_j}\right)^{3/2} \prod_{j=1}^{N-1} \mathrm{d}^3 \boldsymbol{r}_j \tag{2.5}$$

where $\boldsymbol{r}' = \boldsymbol{r}_0$; $\boldsymbol{r}'' = \boldsymbol{r}_N$; $\tau_j = t_j - t_{j-1} = \tau/N = (t'' - t')/N$, and S_j is the short-time action for the free particle,

$$S_j = \int_{t_{j-1}}^{t_j} \tfrac{1}{2} \mu \dot{\boldsymbol{r}}^2 \, \mathrm{d}t = \frac{\mu (\Delta \boldsymbol{r}_j)^2}{2\tau_j}. \tag{2.6}$$

From (2.2) and (2.3), the short-time action (2.6) can be written in the form

$$\begin{aligned} S_j &= \frac{\mu}{2\tau_j}[(\Delta x_j)^2 + (\Delta y_j)^2 + (\Delta z_j)^2] \\ &= \frac{\mu a^2}{(\cosh \eta_j - \cos \xi_j)(\cosh \eta_{j-1} - \cos \xi_{j-1}) \tau_j} \\ &\qquad \times (\cosh \eta_j \cosh \eta_{j-1} - \sinh \eta_j \sinh \eta_{j-1} \cos \Delta\phi_j - \cos \Delta\xi_j). \end{aligned} \tag{2.7}$$

With the propagator in toroidal coordinates, (2.5) and (2.7), we next consider the presence of a ring-shaped defect in three-dimensional space.

3. Particle in regions with a ring-shaped defect

An impenetrable ring in three-dimensional space can be described by a toroidal surface $\eta = \bar{\eta} =$ constant. In this space, various paths of the particle going from an initial point $\boldsymbol{r}'$ to a point $\boldsymbol{r}''$ become homotopically inequivalent and are characterized by different winding numbers n. The propagator (2.5) is expressed as in (1.1), with the short-time action (2.7) modified to

$$S_j = \int_{t_{j-1}}^{t_j} \tfrac{1}{2}\mu\dot{\boldsymbol{r}}^2\,\mathrm{d}t + S_j^{\mathrm{c}} = \frac{\mu(\Delta\boldsymbol{r}_j)^2}{2\tau_j} + S_j^{\mathrm{c}} \tag{3.1}$$

where the term S_j^{c} takes into account the ring's impenetrability. This constraint action wipes out the propagator for the particle hitting the impenetrable ring barrier. Clearly,

$$\exp\left(\frac{\mathrm{i}}{\hbar}S_j^{\mathrm{c}}\right) = \theta(\bar{\eta} - \eta_j) \tag{3.2}$$

where $\theta(\bar{\eta} - \eta_j)$ is a step function with values of unity for $\eta_j < \bar{\eta}$ (outside the ring) and zero for $\eta_j > \bar{\eta}$ (inside the ring).

Let us next consider the case of the particle propagator for winding number $n = 0$. From (2.5), (2.7) and (3.2), we have

$$K_{n=0}(\boldsymbol{r}'', \boldsymbol{r}'; \tau) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left(\frac{\mathrm{i}}{\hbar}S_j^0\right)\theta(\bar{\eta} - \eta_j)\left(\frac{\mu}{2\pi\mathrm{i}\hbar\tau_j}\right)^{3/2} \prod_{j=1}^{N-1} \frac{a^3\sinh\eta_j\,\mathrm{d}\eta_j\,\mathrm{d}\xi_j\,\mathrm{d}\phi_j}{(\cosh\eta_j - \cos\xi_j)^3} \tag{3.3}$$

where

$$S_j^0 = \frac{\mu a^2}{(\cosh\eta_j - \cos\xi_j)(\cosh\eta_{j-1} - \cos\xi_{j-1})\tau_j} \times (\cosh\eta_j\cosh\eta_{j-1} - \sinh\eta_j\sinh\eta_{j-1}\cos\Delta\phi_j - \cos\Delta\xi_j). \tag{3.4}$$

We are now set to perform integration over ξ, ϕ and η.

3.1. Integrating the ξ-part

To integrate the angle ξ, we first apply in $K_{n=0}(\boldsymbol{r}'', \boldsymbol{r}'; \tau)$ a local time rescaling [9] (see appendix 1) of the form

$$\sigma_j = \tau_j(\cosh\eta_j - \cos\xi_j)(\cosh\eta_{j-1} - \cos\xi_{j-1}) \tag{3.5}$$

resulting in the equivalent path integral

$$K_{n=0}(\boldsymbol{r}'', \boldsymbol{r}'; \sigma) = [(\cosh\eta'' - \cos\xi'')(\cosh\eta' - \cos\xi')]^{3/2} \times \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left(\frac{\mathrm{i}}{\hbar}S_j^0\right)\theta(\bar{\eta} - \eta_j)\left(\frac{\mu}{2\pi\mathrm{i}\hbar\sigma_j}\right)^{3/2} \prod_{j=1}^{N-1}(a^3\sinh\eta_j\,\mathrm{d}\eta_j\,\mathrm{d}\xi_j\,\mathrm{d}\phi_j) \tag{3.6}$$

where

$$S_j^0 = \frac{\mu a^2}{\sigma_j}(\cosh\eta_j\cosh\eta_{j-1} - \sinh\eta_j\sinh\eta_{j-1}\cos\Delta\phi_j - \cos\Delta\xi_j). \tag{3.7}$$

A Taylor series expansion of $\cos\Delta\xi_j$ in (3.7) is then performed up to the fourth-order term:

$$S_j^0 \simeq \frac{\mu a^2}{\sigma_j}\left(\cosh\eta_j\cosh\eta_{j-1} - \sinh\eta_j\sinh\eta_{j-1}\cos\Delta\phi_j - 1 + \frac{(\Delta\xi_j)^2}{2} - \frac{(\Delta\xi_j)^4}{24}\right). \tag{3.8}$$

Note that $[(\Delta\xi_j)^4/\sigma_j] = O(\sigma_j)$, and hence still contributes significantly to the path integral. Once exponentiated within the path integral, this term can be handled using the formula [13]

$$\int_{-\infty}^{\infty}\exp(-\alpha x^2 + \beta x^4)\,\mathrm{d}x = \int_{-\infty}^{\infty}\exp\left(-\alpha x^2 + \frac{3\beta}{4\alpha^2}\right)\mathrm{d}x \tag{3.9}$$

valid for $\mathrm{Re}\,\alpha > 0$, $|\alpha|$ large, which can be satisfied by adding a small imaginary mass $\tilde{\mu}$ to μ in S_j^0. The limit $\tilde{\mu} \to 0$ is then taken after the calculations. This leads to

$$\begin{aligned} K_{n=0}(\boldsymbol{r}'', \boldsymbol{r}'; \sigma) &= [(\cosh\eta'' - \cos\xi'')(\cosh\eta' - \cos\xi')]^{3/2} \\ &\quad\times \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left(\frac{\mathrm{i}\mu a^2}{2\hbar\sigma_j}(\Delta\xi_j)^2\right. \\ &\quad\left. - \frac{\mathrm{i}\mu a^2}{\hbar\sigma_j}(1 - \cosh\eta_j\cosh\eta_{j-1} + \sinh\eta_j\sinh\eta_{j-1}\cos\Delta\phi_j) + \frac{\mathrm{i}\hbar\sigma_j}{8\mu a^2}\right) \\ &\quad\times\theta(\bar{\eta} - \eta_j)\left(\frac{\mu}{2\pi\mathrm{i}\hbar\sigma_j}\right)^{3/2}\prod_{j=1}^{N-1}(a^3\sinh\eta_j\,\mathrm{d}\eta_j\,\mathrm{d}\xi_j\,\mathrm{d}\phi_j). \end{aligned} \tag{3.10}$$

The ξ-angular path integral can be extracted from (3.10) and is of the form

$$A_\xi(\xi'', \xi'; \sigma) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left(\frac{\mathrm{i}}{\hbar}S_j(\xi)\right)\left(\frac{-\mathrm{i}\mu a^2}{2\pi\hbar\sigma_j}\right)^{1/2}\prod_{j=1}^{N-1}\mathrm{d}\xi_j \tag{3.11}$$

where $S_j(\xi) = (\mu a^2/2\sigma_j)(\Delta\xi_j)^2$. Observing that (3.11) is similar in form to the Gaussian path integral for a free particle along the ξ-coordinate, integration is readily performed, to yield

$$A_\xi(\xi'', \xi'; \sigma) = \left(\frac{\mu a^2}{2\pi\mathrm{i}\hbar\sigma}\right)^{1/2}\exp\left(\frac{\mathrm{i}\mu a^2(\xi'' - \xi')^2}{2\hbar\sigma}\right). \tag{3.12}$$

With (3.12), equation (3.10) can be written as

$$K_{n=0}(\boldsymbol{r}'', \boldsymbol{r}'; \sigma) = A_\xi(\xi'', \xi'; \sigma)\,Q(\eta'', \phi'', \eta', \phi'; \sigma) \tag{3.13}$$

where

$$\begin{aligned} Q(\eta'', \phi'', \eta', \phi'; \sigma) &= \frac{2\pi}{a}[(\cosh\eta'' - \cos\xi'')(\cosh\eta' - \cos\xi')]^{3/2} \\ &\quad\times\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left(\frac{\mathrm{i}}{\hbar}S_j(\eta, \phi)\right)\theta(\bar{\eta} - \eta_j)\left(\frac{\mu}{2\pi\mathrm{i}\hbar\sigma_j}\right)\prod_{j=1}^{N-1}(a^2\sinh\eta_j\,\mathrm{d}\eta_j\,\mathrm{d}\phi_j) \end{aligned} \tag{3.14}$$

and

$$S_j(\eta, \phi) = \frac{\hbar^2\sigma_j}{8\mu a^2} - \frac{\mu a^2}{\sigma_j}(1 - \cosh\eta_j\cosh\eta_{j-1} + \sinh\eta_j\sinh\eta_{j-1}\cos\Delta\phi_j). \tag{3.15}$$

This brings us to path integration of the η- and ϕ-variables.

3.2. Integrating the ϕ-part

To facilitate integration over ϕ_j, we use the relation

$$\begin{aligned}\exp(z\cos\Delta\phi_j) &= \sum_{m_j=-\infty}^{\infty}\exp(\mathrm{i}m_j\Delta\phi_j)I_{m_j}(z)\\ &\simeq (2\pi z)^{-1/2}\sum_{m_j=-\infty}^{\infty}\exp(\mathrm{i}m_j\Delta\phi_j)\exp\left(z-\frac{(m^2-\frac{1}{4})}{2z}\right)\end{aligned} \tag{3.16}$$

in which the large argument asymptotic expansion of the modified Bessel function of the first kind, $I_m(z)$, can be applied with $z=-(\mathrm{i}\mu a^2/\hbar\sigma_j)\sinh\eta_j\sinh\eta_{j-1}$, $\sigma_j\to 0$ as $N\to\infty$. Substitution gives

$$\begin{aligned}Q(\eta'',\phi'',\eta',\phi';\sigma) &= \frac{2\pi}{a}[(\cosh\eta''-\cos\xi'')(\cosh\eta'-\cos\xi')]^{3/2}\\ &\times\lim_{N\to\infty}\int\prod_{j=1}^{N}\sum_{m_j=-\infty}^{\infty}\exp(\mathrm{i}m_j\Delta\phi_j)\exp\left(\frac{\mathrm{i}\hbar\sigma_j}{8\mu a^2}-\frac{\mathrm{i}\mu a^2}{\hbar\sigma_j}\right.\\ &\left.+\frac{\mathrm{i}\mu a^2}{\hbar\sigma_j}(\cosh\eta_j\cosh\eta_{j-1}-\sinh\eta_j\sinh\eta_{j-1})+\frac{(m_j^2-\frac{1}{4})\hbar\sigma_j}{2\mathrm{i}\mu a^2\sinh\eta_j\sinh\eta_{j-1}}\right)\\ &\times\left(\frac{\mathrm{i}\hbar\sigma_j}{2\pi\mu a^2\sinh\eta_j\sinh\eta_{j-1}}\right)^{1/2}\theta(\bar{\eta}-\eta_j)\left(\frac{\mu}{2\pi\mathrm{i}\hbar\sigma_j}\right)\prod_{j=1}^{N-1}(a^2\sinh\eta_j\,\mathrm{d}\eta_j\,\mathrm{d}\phi_j).\end{aligned} \tag{3.17}$$

With the above result, the ϕ-angular part can now be readily integrated, using

$$\delta_{mm'}=\frac{1}{2\pi}\int_0^{2\pi}\exp[\mathrm{i}(m'-m)\phi]\,\mathrm{d}\phi \tag{3.18}$$

which yields

$$\begin{aligned}Q(\eta'',\phi'',\eta',\phi';\sigma) &= \frac{[(\cosh\eta''-\cos\xi'')(\cosh\eta'-\cos\xi')]^{3/2}}{a^2(\sinh\eta''\sinh\eta')^{1/2}}\\ &\times\sum_{m=-\infty}^{\infty}\exp[\mathrm{i}m(\phi''-\phi')]\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left(\frac{-\mathrm{i}\mu a^2}{\hbar\sigma_j}(1-\cosh\Delta\eta_j)\right.\\ &\left.+\frac{(m^2-\frac{1}{4})\hbar\sigma_j}{2\mathrm{i}\mu a^2\sinh\eta_j\sinh\eta_{j-1}}+\frac{\mathrm{i}\hbar\sigma_j}{8\mu a^2}\right)\theta(\bar{\eta}-\eta_j)\left(\frac{\mu}{2\pi\mathrm{i}\hbar\sigma_j}\right)^{1/2}\prod_{j=1}^{N-1}(a\,\mathrm{d}\eta_j).\end{aligned} \tag{3.19}$$

At this point, only the η-part remains to be path integrated.

3.3. *Integrating the η-part*

Taking $\sigma_j \to -\sigma_j$ and rearranging terms in the last expression, (3.19) can be rewritten as

$$Q(\eta'', \phi'', \eta', \phi'; \sigma) = \frac{[(\cosh\eta'' - \cos\xi'')(\cosh\eta' - \cos\xi')]^{3/2}}{a^2(\sinh\eta''\sinh\eta')^{1/2}} \exp\left(\frac{-i\hbar\sigma}{8\mu a^2}\right) \times \sum_{m=-\infty}^{\infty} \exp[im(\phi'' - \phi')]\hat{K}(\eta'', \eta'; \sigma) \tag{3.20}$$

where $\hat{K}(\eta'', \eta'; \sigma)$ is a one-dimensional path integral in η given by

$$\hat{K}(\eta'', \eta'; \sigma) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left(\frac{i}{\hbar} S_j(\eta)\right) \theta(\bar{\eta} - \eta_j) \left(\frac{\mu i}{2\pi\hbar\sigma_j}\right)^{1/2} \prod_{j=1}^{N-1} (a\, d\eta_j) \tag{3.21}$$

with the short-time action

$$S_j(\eta) = \frac{\mu a^2}{\sigma_j}(1 - \cosh\Delta\eta_j) + \frac{(m^2 - \frac{1}{4})\hbar^2\sigma_j}{2\mu a^2 \sinh\eta_j \sinh\eta_{j-1}}. \tag{3.22}$$

A further simplification can be achieved by using the relations

$$\cosh\Delta\eta_j = 4\cosh\left(\frac{\Delta\eta_j}{2}\right) + \frac{(\Delta\eta_j/2)^4}{2} - 3 \tag{3.23}$$

and

$$\frac{-\hbar\sigma_j(m^2 - \frac{1}{4})}{2i\mu a^2 \sinh\eta_j \sinh\eta_{j-1}} \simeq \frac{-\hbar\sigma_j(m^2 - \frac{1}{4})}{8i\mu a^2 \sinh(\eta_j/2)\sinh(\eta_{j-1}/2)} + \frac{\hbar\sigma_j(m^2 - \frac{1}{4})}{8i\mu a^2 \cosh(\eta_j/2)\cosh(\eta_{j-1}/2)}. \tag{3.24}$$

Note that in (3.24), terms of order $\sigma^{1+\varepsilon}(\varepsilon > 0)$ have been dropped, making no significant contribution to the path integral. Again, the fourth-order term in (3.23) can be handled within the path integral by using (3.9), i.e. the equivalent action, when exponentiated in the path integral, becomes

$$\exp\left[\frac{i}{\hbar} S_j(\eta)\right] = \exp\left[\frac{4i\mu a^2}{\hbar\sigma_j}\left(1 - \cosh\frac{\Delta\eta_j}{2}\right) + \frac{3i\hbar\sigma_j}{32\mu a^2} - \frac{\hbar\sigma_j(m^2 - \frac{1}{4})}{8i\mu a^2 \sinh(\eta_j/2)\sinh(\eta_{j-1}/2)} + \frac{\hbar\sigma_j(m^2 - \frac{1}{4})}{8i\mu a^2 \cosh(\eta_j/2)\cosh(\eta_{j-1}/2)}\right]. \tag{3.25}$$

The propagator $\hat{K}(\eta'', \eta'; \sigma)$ in (3.21) can now be written as

$$\hat{K}(\eta'', \eta'; \sigma) = \lim_{N\to\infty} \frac{1}{2} \int \prod_{j=1}^{N} \exp\left(\frac{i}{\hbar}\hat{S}_j\right) \theta(\bar{\eta} - \eta_j) \left(\frac{4\mu i}{2\pi\hbar\sigma_j}\right)^{1/2} \prod_{j=1}^{N-1} \left(\frac{a}{2}\, d\eta_j\right) \tag{3.26}$$

where

$$\hat{S}_j = \frac{4\mu a^2}{\sigma_j}\left(1-\cosh\frac{\Delta\eta_j}{2}\right) + \frac{\hbar^2\sigma_j}{8\mu a^2}\left(\frac{m^2-\frac{1}{4}}{\sinh(\eta_j/2)\sinh(\eta_{j-1}/2)} - \frac{m^2-\frac{1}{4}}{\cosh(\eta_j/2)\cosh(\eta_{j-1}/2)}\right) + \frac{3\hbar^2\sigma_j}{32\mu a^2}. \tag{3.27}$$

Equation (3.26), with (3.27), is formally identical to the propagator of the one-dimensional modified Poschl–Teller oscillator whose path integral, although non-Gaussian in form, has been evaluated through the application of group theory [10]. The procedure takes advantage of the dynamical symmetry of the associated Lagrangian and consists of expanding the short-time propagator in matrix elements of the unitary irreducible representations (UIRs) of the symmetry group (SU(1,1) in this case). By identifying the coordinates with the group parameters, the path integral is transformed into an integral over the SU(1,1) group manifold—which is then simply carried out using the orthogonality of the representations (see appendix 2). The result of this procedure gives for $Q(\eta'',\phi'';\eta',\phi';\sigma)$ in (3.20) the expression

$$Q(\eta'',\phi'';\eta',\phi';\sigma) = \frac{[(\cosh\eta''-\cos\xi'')(\cosh\eta'-\cos\xi')]^{3/2}}{2a(2\pi a)^2} \times \sum_{m=-\infty}^{\infty}\exp[\mathrm{i}m(\phi''-\phi')]\int_0^\infty \mathrm{d}\rho\, 2\rho\tanh(\pi\rho)d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta'')d_{m0}^{-(1/2)+\mathrm{i}\rho,0*}(\eta') \times\exp\left[-\frac{\mathrm{i}}{\hbar}\left(\frac{(\rho^2+\frac{1}{16})\hbar^2}{2\mu a^2}\right)\sigma\right] \tag{3.28}$$

where, $d_{ab}^{l,\lambda}(\eta)$ are the Bargmann functions [14]. Note that, in carrying out the integration using the orthogonality relation (A2.10) of the Bargmann functions, we take the ring to be infinitesimally thin, i.e. $\bar{\eta}\to\infty$, in (3.2). Using these results, we now write (3.13) as

$$K_{n=0}(\boldsymbol{r}'',\boldsymbol{r}';\sigma) = \frac{[(\cosh\eta''-\cos\xi'')(\cosh\eta'-\cos\xi')]^{3/2}}{4\pi a^2} \times \sum_{m=-\infty}^{\infty}\exp[\mathrm{i}m(\phi''-\phi')]\int_0^\infty \mathrm{d}\rho\, 2\rho\tanh(\pi\rho)d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta'')d_{m0}^{-(1/2)+\mathrm{i}\rho,0*}(\eta') \times\exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{(\rho^2+\frac{1}{16})\hbar^2}{2\mu a^2}\right)\sigma\right]\left(\frac{\mu}{2\pi\mathrm{i}\hbar\sigma}\right)^{1/2}\exp\left[\frac{\mathrm{i}\mu a^2}{2\hbar\sigma}(\xi''-\xi')^2\right]. \tag{3.29}$$

Here, we have transformed back to $\sigma\to-\sigma$.

The replacement of $(\xi''-\xi')$ by $(\xi''-\xi'+2\pi n)$ in (3.29) gives the nth winding propagator. From (1.1), all such propagators corresponding to winding number n must be summed to give the total propagator:

$$K(\boldsymbol{r}'',\boldsymbol{r}';\sigma) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{[(\cosh\eta''-\cos\xi'')(\cosh\eta'-\cos\xi')]^{3/2}}{4\pi a^2}$$

$$\times \exp[\mathrm{i}m(\phi'' - \phi')] \int_0^\infty \mathrm{d}\rho\, 2\rho \tanh(\pi\rho) d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta'') d_{m0}^{-(1/2)+\mathrm{i}\rho,0*}(\eta')$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{(\rho^2 + \frac{1}{16})\hbar^2}{2\mu a^2}\right)\sigma\right]\left(\frac{\mu}{2\pi\mathrm{i}\hbar\sigma}\right)^{1/2} \exp\left[\frac{\mathrm{i}\mu a^2}{2\hbar\sigma}(\xi'' - \xi' + 2\pi n)^2\right]. \tag{3.30}$$

Here, the winding number n signifies, if positive, a particle's path that first goes inside the ring and then loops n times around it; and, if negative, a path that first passes outside the ring and then loops $|n| - 1$ times (see figure 1). For instance, the winding numbers $n = -1$ and $n = 0$ correspond to paths passing outside and inside the ring, respectively, but with no winding. We note that if the ring radius goes to infinity, a ring solenoid becomes a cylinder in three-dimensional space. When symmetrically sectioned, the cylinder leads to a singularity in a plane previously discussed using polar coordinates [4–7] where the polar angle θ ($0 \leqslant \theta < 2\pi$) replaces the role of the periodic ξ variable ($0 \leqslant \xi < 2\pi$) in toroidal coordinates.

The full propagator can also be put in another convenient form after noticing that the last two factors in (3.30) can be expressed as an integral in a parameter γ ($-\infty < \gamma < \infty$),

$$\left(\frac{\mu}{2\pi\mathrm{i}\hbar\sigma}\right)^{1/2} \exp\left[\frac{\mathrm{i}\mu a^2}{2\hbar\sigma}(\xi'' - \xi' + 2\pi n)^2\right]$$
$$= \frac{1}{2\pi a}\int_{-\infty}^{\infty} \mathrm{d}\gamma \exp\left[\mathrm{i}\gamma(\xi'' - \xi' + 2\pi n) - \frac{\mathrm{i}\hbar\gamma^2\sigma}{2\mu a^2}\right] \tag{3.31}$$

leading to

$$K(r'', r'; \sigma) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{[(\cosh\eta'' - \cos\xi'')(\cosh\eta' - \cos\xi')]^{3/2}}{8\pi^2 a^3} \exp[\mathrm{i}m(\phi'' - \phi')]$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}\gamma \int_0^{\rho} \mathrm{d}\rho\, 2\rho \tanh(\pi\rho) d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta'') d_{m0}^{-(1/2)+\mathrm{i}\rho,0*}(\eta')$$
$$\times \exp[\mathrm{i}\gamma(\xi'' - \xi' + 2\pi n)] \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{(\rho^2 - \gamma^2 + \frac{1}{16})}{2\mu a^2}\right)\hbar^2\sigma\right]. \tag{3.32}$$

Equation (3.32), which is a winding number representation of the total propagator, can further be recast using Poisson's sum formula,

$$\sum_{n=-\infty}^{\infty} \exp(2\pi\mathrm{i}n\gamma) = \sum_{s=-\infty}^{\infty} \delta(\gamma - s) \tag{3.33}$$

where the propagator becomes

$$K(r'', r'; \sigma) = \sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} \frac{[(\cosh\eta'' - \cos\xi'')(\cosh\eta' - \cos\xi')]^{3/2}}{8\pi^2 a^3} \exp[\mathrm{i}m(\phi'' - \phi')]$$
$$\times \int_0^{\rho} \mathrm{d}\rho\, 2\rho \tanh(\pi\rho) d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta'') d_{m0}^{-(1/2)+\mathrm{i}\rho,0*}(\eta')$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}\gamma\, \delta(\gamma - s) \exp[\mathrm{i}\gamma(\xi'' - \xi')] \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{(\rho^2 - \gamma^2 + \frac{1}{16})}{2\mu a^2}\right)\hbar^2\sigma\right]. \tag{3.34}$$

With the δ-function, the γ-integration can be carried out, and the propagator can be written in the form

$$K(\boldsymbol{r}'', \boldsymbol{r}'; \sigma) = \sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \int_0^{\rho} \mathrm{d}\rho\, \Psi^*_{\rho s m}(\eta', \xi', \phi') \Psi_{\rho s m}(\eta'', \xi'', \phi'') \times \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{(\rho^2 - s^2 + \frac{1}{16})}{2\mu a^2}\right)\hbar^2\sigma\right] \tag{3.35}$$

where $\Psi(\eta, \xi, \phi)$ is given by

$$\Psi_{\rho s m}(\eta, \xi, \phi) = \frac{(\cosh\eta - \cos\xi)^{3/2}}{2\sqrt{2\pi}a}\sqrt{2\rho\tanh(\pi\rho)}d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta)\exp(\mathrm{i}m\phi)\exp(\mathrm{i}s\xi). \tag{3.36}$$

Note that the propagator given in (3.35) is parametrized by the rescaled time σ related to τ as given by (4.2). It would then be convenient to consider the energy Green function.

4. The energy Green function

The energy Green function is obtained as the Fourier transform of the propagator

$$\begin{aligned} G(\boldsymbol{r}'', \boldsymbol{r}'; E) &= (\mathrm{i}\hbar)^{-1}\int_0^{\infty} K(\boldsymbol{r}'', \boldsymbol{r}'; \tau)\exp(\mathrm{i}E\tau/\hbar)\,\mathrm{d}\tau \\ &= (\mathrm{i}\hbar)^{-1}\int_0^{\infty} \exp[\mathrm{i}E\sigma/\hbar(\cosh\eta'' - \cos\xi'')(\cosh\eta' - \cos\xi')] \\ &\quad \times K(\boldsymbol{r}'', \boldsymbol{r}'; \sigma)(\mathrm{d}\tau/\mathrm{d}\sigma)\,\mathrm{d}\sigma \end{aligned} \tag{4.1}$$

in which the global rescaled time for the case being considered is given by

$$\sigma = \tau(\cosh\eta'' - \cos\xi'')(\cosh\eta' - \cos\xi'). \tag{4.2}$$

Integration over σ yields

$$G(\boldsymbol{r}'', \boldsymbol{r}'; E) = \sum_{n=-\infty}^{\infty} G_n(\boldsymbol{r}'', \boldsymbol{r}'; E) \tag{4.3}$$

where, with (3.32), we have

$$\begin{aligned} G_n(\boldsymbol{r}'', \boldsymbol{r}'; E) = &\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\gamma\int_0^{\infty}\mathrm{d}\rho\frac{[(\cosh\eta'' - \cos\xi'')(\cosh\eta' - \cos\xi')]^{3/2}}{\mathrm{i}\hbar 8\pi^2 a^3}2\rho\tanh(\pi\rho) \\ &\times \exp[\mathrm{i}m(\phi'' - \phi')]d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta'')d_{m0}^{-(1/2)+\mathrm{i}\rho,0,}(\eta')\exp[\mathrm{i}\gamma(\xi'' - \xi' + 2\pi n)] \\ &\times \left[\frac{E}{\hbar} + \left((\rho^2 - \gamma^2 + \tfrac{1}{16})(\cosh\eta'' - \cos\xi'')\frac{(\cosh\eta' - \cos\xi')\hbar}{2\mu a^2}\right) + \mathrm{i}\varepsilon\right]^{-1}_{\varepsilon\to 0}. \end{aligned} \tag{4.4}$$

This gives the Green function in closed form for a particle moving in a region with a ring-shaped toplogical defect. An alternative form can also be obtained by again using (3.33), which allows one to integrate γ. This yields

$$G(r'', r'; E) = \sum_{s=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_0^{\infty} \mathrm{d}\rho \frac{[(\cosh \eta'' - \cos \xi'')(\cosh \eta' - \cos \xi')]^{3/2}}{\mathrm{i}\hbar 8\pi^2 a^3}$$
$$\times \exp[\mathrm{i}m(\phi''-\phi')] \exp[\mathrm{i}s(\xi''-\xi')] 2\rho \tanh(\pi\rho) d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta'') d_{m0}^{-(1/2)+\mathrm{i}\rho,0*}(\eta')$$
$$\times \left[\frac{E}{\hbar} + \left((\rho^2 - s^2 + \tfrac{1}{16})(\cosh \eta'' - \cos \xi'')\frac{(\cosh \eta' - \cos \xi')\hbar}{2\mu a^2}\right) + \mathrm{i}\varepsilon\right]^{-1}_{\varepsilon \to 0} . \tag{4.5}$$

5. Application: the toroidal Aharonov–Bohm effect

For magnetic flux enclosed in a toroid, the vector potential $\boldsymbol{A}$, in spherical coordinates, is an infinite series in Legendre functions. In contrast, the vector potential $\boldsymbol{A}$ in toroidal coordinates becomes quite simple, taking the form, $\boldsymbol{A} = (\Phi_0/2\pi)\boldsymbol{\nabla}\boldsymbol{\xi}$. Here, Φ_0 is the enclosed flux, and ξ is the coordinate associated with the spherical surfaces. The vector potential lines are in fact toroidal in form and orthogonal to the ξ-surfaces. This reminds us of the case of the solenoidal vector potential $\boldsymbol{A} = (\Phi_0/2\pi)\boldsymbol{\nabla}\theta$, where θ is the polar angle.

The corresponding short-time action can then be expressed as

$$S_j = \int_{t_{j-1}}^{t_j} \left(\frac{1}{2}\mu \dot{r}^2 + \frac{e}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{r}}\right) \mathrm{d}t = \frac{\mu}{2\tau_j}(\Delta \boldsymbol{r}_j)^2 + g\hbar\Delta\xi_j \tag{5.1}$$

where $g = e\Phi_0/2\pi\hbar c$. Following the procedure discussed in section 3, the interaction term, when exponentiated with $\exp[(\mathrm{i}/\hbar)S_j]$, gives rise to the expression

$$\prod_{j=1}^{N} \exp(\mathrm{i}g\Delta\xi_j) \to \exp[\mathrm{i}g(\xi'' - \xi' + 2\pi n)]. \tag{5.2}$$

In (5.2), the winding number n comes from the replacement, $\xi'' - \xi' \to \xi'' - \xi' + 2\pi n$. Thus, the resulting propagator for the system with magnetic flux confined in a ring is

$$K(r'', r'; \sigma) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{[(\cosh \eta'' - \cos \xi'')(\cosh \eta' - \cos \xi')]^{3/2}}{4\pi a^2} \exp[\mathrm{i}m(\phi'' - \phi')]$$
$$\times \int_0^{\infty} \mathrm{d}\rho\, 2\rho \tanh(\pi\rho) d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta'') d_{m0}^{-(1/2)+\mathrm{i}\rho,0*}(\eta')$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{\rho^2 + \frac{1}{16}}{2\mu a^2}\right)\hbar^2\sigma\right]\left(\frac{\mu}{2\pi\mathrm{i}\hbar\sigma}\right)^{1/2}$$
$$\times \exp\left(\frac{\mathrm{i}\mu a^2}{2\hbar\sigma}(\xi'' - \xi' + 2\pi n)^2 + \mathrm{i}g(\xi'' - \xi' + 2\pi n)\right). \tag{5.3}$$

This can also be written in the same way as (3.32) as

$$K(r'',r';\sigma)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{[(\cosh\eta''-\cos\xi'')(\cosh\eta'-\cos\xi')]^{3/2}}{8\pi^2a^3}\exp[\mathrm{i}m(\phi''-\phi')]$$
$$\times\int_{-\infty}^{\infty}\mathrm{d}\gamma\int_0^{\infty}\mathrm{d}\rho\,2\rho\tanh(\pi\rho)d_{m0}^{-(1/2)+\mathrm{i}\rho,0*}(\eta')d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta'')$$
$$\times\exp[\mathrm{i}(\gamma+g)(\xi''-\xi'+2\pi n)]\exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{\rho^2-\gamma^2+\frac{1}{16}}{2\mu a^2}\right)\hbar^2\sigma\right].\qquad(5.4)$$

It is also interesting to look at the interference terms associated with two electron paths characterized by different winding numbers n and l about the toroidal flux. The interference terms are given by

$$K_n^*K_l+K_l^*K_n=\left[\frac{(\cosh\eta''-\cos\xi'')^3(\cosh\eta'-\cos\xi')^3}{(4\pi a^2)^2}\left(\frac{\mu}{2\pi\hbar\sigma}\right)\right]$$
$$\times\sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty}\exp[\mathrm{i}(m-m')(\phi''-\phi')]$$
$$\times\int_0^{\infty}\int_0^{\infty}\mathrm{d}\rho\,\mathrm{d}\rho'\,2\rho 2\rho'\tanh(\pi\rho)\tanh(\pi\rho')$$
$$\times d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta'')d_{m'0}^{-(1/2)+\mathrm{i}\rho',0}(\eta'')d_{m0}^{-(1/2)+\mathrm{i}\rho,0*}(\eta')d_{m'0}^{-(1/2)+\mathrm{i}\rho',0*}(\eta')$$
$$\times 2\cos\left(2\pi(l-n)g+\frac{\mu a^2}{2\hbar\sigma}\left[(\xi''-\xi'+2\pi l)^2-(\xi''-\xi'+2\pi n)^2\right]\right).\ (5.5)$$

This shows that the interference terms are proportional to a flux-dependent factor, i.e.

$$K_n^*K_l+K_l^*K_n\propto 2\cos\left[2\pi(l-n)\left(g+\frac{\mu a^2}{\hbar\sigma}\bar{\xi}\right)+2\pi^2\left(\frac{\mu a^2}{\hbar\sigma}\right)(l-n)(l+n+1)\right]\quad(5.6)$$

where we have defined, $\bar{\xi}\equiv(\xi''-\xi'+\pi)$ (see figure 1). In terms of τ the last expression is given by

$$K_n^*K_l+K_l^*K_n\propto 2\cos\{2\pi(l-n)[g+(\mu a^2\bar{\xi}/\hbar\tau(\cosh\eta''-\cos\xi'')(\cosh\eta'-\cos\xi'))]$$
$$+2\pi^2\mu a^2(l-n)(l+n+1)/\hbar\tau(\cosh\eta''-\cos\xi'')(\cosh\eta'-\cos\xi')\}.\quad(5.7)$$

The interference pattern in (5.7) depends on the following factors: the flux enclosed, the winding numbers, the size of the ring, as well as the ξ', η' and ξ'', η'' coordinates of the source and detector, respectively. For paths which do not wind around the toroid, e.g. for $l=-1$ and $n=0$, what remains is the usual flux-dependent shift of the AB effect, i.e.

$$K_0^*K_1+K_1^*K_0\propto\cos(2\pi g)\qquad(5.8)$$

at $\bar{\xi}=0$, where $g=e\Phi_0/2\pi\hbar c$ and Φ_0 is the magnetic flux. In addition, (5.7) contains the term $(l+n+1)$, dependent on higher winding numbers. This contribution disappears when $l+n+1=0$, as in the case of $l=0$, and $n=-1$ (no winding), wherein the interference shift is due mainly to the usual AB effect. These shifts do not appear in present AB experiments, where apparently only electron paths with no winding are generated. Furthermore, once a summation over all n is performed to obtain the full propagator, the shifts for paths with higher winding numbers may be smeared out and consequently disappear.

6. Conclusion

In this paper, the toroidal path integral for a particle moving in a space with a ring-shaped topological defect has been evaluated in closed form. In this multiply connected space, the full propagator is taken as a sum of partial propagators corresponding to homotopically inequivalent paths characterized by the number of times they wind about the ring. The partial propagators were evaluated exactly with the aid of techniques used in handling non-Gaussian path integrals, namely local time rescaling and application of results of path integration over the SU(1,1) group manifold. Furthermore, the Fourier transform of the propagator yielded the energy Green function in closed form.

As a first application, the problem of an electron path winding about an infinitesimally thin ring of magnetic flux was also considered. This corresponds to the Tonomura experiment [11], which gave a definitive verification of the AB effect. The interference pattern was evaluated and found to consist of the usual flux-dependent AB shift and an additional purely topological phase dependent on the winding number and the size of the ring. This result has also been found in the case of the solenoidal AB effect. [7].

An extension of this work involves the explicit calculation of the scattering matrix for a particle in a region with confined toroidal flux using earlier methods [15]. This will be compared with other results such as those of Lyuboshitz and Smorodinski [16] calculated using eikonal and Born approximations. It would also be interesting to find applications in the problem of electron transport and localization in microscopic, quantum-mechanical structures [17].

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Appendix 1

The technique of local time rescaling wherein paths are reparametrized by a space-dependent time has been crucial in allowing the exact evaluation of various non-Gaussian path integrals. The mathematical basis for this has been given by Fischer, Leschke and Muller [9]. Discussions and applications of this technique abound in recent literature [18].

The equivalence of local time rescaling to transformations applied to the Schrödinger equation to reduce it to a solvable form has also been discussed [19]. Consider the time-dependent Schrödinger equation

$$[(-\hbar^2/2m)\partial_x^2 + V(x,t)]\Psi(x,t) = i\hbar\partial_t\Psi(x,t) \tag{A1.1}$$

which may not be in a directly solvable form. Under an appropriate transformation, say $x = f(z)$, the wavefunction is taken in the form $\Psi(x,t) = g(z)F(z,s)$ such that $F(z,s)$ satisfies a separable solvable differential equation of the form

$$[(-\hbar^2/2m)\partial_z^2 + \hat{V}(z,s)]F(z,s) = i\hbar\partial_s F(z,s). \tag{A1.2}$$

The requirement imposed by (A1.2) gives $g(z) = (\mathrm{d}f/\mathrm{d}z)^{1/2}$ and $s = t(\mathrm{d}f/\mathrm{d}z)^2$.

We can thus find the correspondence of the path-integral solution presented in this paper and the free particle Schrödinger equation in toroidal coordinates given by

$$-\frac{\hbar^2}{2m}h^{-3}(\eta,\xi)\left[\frac{1}{\sinh\eta}\frac{\partial}{\partial\eta}\left(h(\eta,\xi)\sinh\eta\frac{\partial}{\partial\eta}\right)+\frac{\partial}{\partial\xi}\left(h(\eta,\xi)\frac{\partial}{\partial\xi}\right)\right.$$
$$\left.+\frac{1}{\sinh^2\eta}h(\eta,\xi)\frac{\partial^2}{\partial\phi^2}\right]\Psi(\eta,\xi,\phi,t)=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(\eta,\xi,\phi,t) \tag{A1.3}$$

where $h(\eta,\xi) = a/(\cosh\eta - \cos\xi)$. In analogy to the above procedure for reducing a differential equation into solvable form, the wavefunction is taken as

$$\Psi(\eta,\xi,\phi,t) = (\cosh\eta-\cos\xi)^{1/2}F(\eta,\xi,\phi,s) \tag{A1.4}$$

where $s = t(\cosh\eta - \cos\xi)^2$. Substitution in (A1.3) and straightforward simplification then yields the separable form

$$\left[\frac{1}{\sinh\eta\partial\eta}\frac{\partial}{\partial\eta}\left(\sinh\eta\frac{\partial}{\partial\eta}\right)+\frac{1}{\sinh^2\eta\partial\phi^2}\frac{\partial^2}{}+\frac{\partial^2}{\partial\xi^2}+(k_s^2a^2+\tfrac{1}{4})\right]F(\eta,\xi,\phi,s)=0 \tag{A1.5}$$

where $k_s^2 = 2mE_s/\hbar^2$, E_s corresponding to $(\mathrm{i}\hbar\partial/\partial s)$. The solution of (A1.3), obtained with the aid of (A1.5), holds when the particle is outside the toroidal excluded region, and vanishes at the impenetrable toroidal boundary.

Appendix 2

Here we briefly present the key steps in transforming the propagator in (3.26) with the action (3.27) into the SU(1,1) path integral and its subsequent evaluation. The presentation follows the procedure given in the work of Bohm and Junker [10].

Using the asymptotic formula valid for large ϵ and integer s,

$$\exp\left(-\frac{(s^2-\frac{1}{4})}{2\epsilon}\right)=\left(\frac{\epsilon}{2\pi}\right)^{1/2}\int_0^{2\pi}\exp[\mathrm{i}sy-\epsilon(1-\cos y)]\,\mathrm{d}y \tag{A2.1}$$

we can exponentiate the third and fourth terms of (3.27) such that

$$\exp\left[\frac{-(m^2-\frac{1}{4})\hbar\sigma_j}{8\mathrm{i}\mu a^2\sinh(\eta_j/2)\sinh(\eta_{j-1}/2)}+\frac{-(m^2-\frac{1}{4})\hbar\sigma_j}{8\mathrm{i}\mu a^2\cosh(\eta_j/2)\cosh(\eta_{j-1}/2)}\right]$$
$$=\frac{\mu a^2}{\pi\hbar\sigma_j}[\sinh(\eta_j)\sinh(\eta_{j-1})]^{1/2}$$
$$\times\int_0^{2\pi}\int_0^{2\pi}\exp\left\{\mathrm{i}m(\alpha_j+\beta_j)+\frac{4\mathrm{i}\mu a^2}{\hbar\sigma_j}\cosh\frac{\Delta\eta_j}{2}-\frac{4\mathrm{i}\mu a^2}{\hbar\sigma_j}\right.$$
$$\left.\times\left[\cosh\left(\frac{\eta_j}{2}\right)\cosh\left(\frac{\eta_{j-1}}{2}\right)\cos\beta_j-\sinh\left(\frac{\eta_j}{2}\right)\sinh\left(\frac{\eta_{j-1}}{2}\right)\cos\alpha_j\right]\right\}\mathrm{d}\alpha_j\,\mathrm{d}\beta_j \tag{A2.2}$$

The angles α_j and β_j may be cast in terms of Euler angles φ_j and ψ_j as $\alpha_j = \frac{1}{2}(\Delta\varphi_j - \Delta\psi_j)$ and $\beta_j = \frac{1}{2}(\Delta\varphi_j + \Delta\psi_j)$. This leads to

$$\exp\left(\frac{\mathrm{i}}{\hbar}\hat{S}_j\right) = \frac{\mu a^2}{2\pi\hbar\sigma_j}(\sinh\eta_j \sinh\eta_{j-1})^{1/2} \times \int_0^{2\pi}\int_{-2\pi}^{2\pi} \exp\left[\frac{4\mathrm{i}\mu a^2}{\hbar\sigma_j}\left(1-\cosh\frac{\Omega_j}{2}\right) + \mathrm{i}m\Delta\varphi_j + \frac{3\mathrm{i}\hbar\sigma_j}{32\mu a^2}\right]\mathrm{d}\varphi_j\,\mathrm{d}\psi_j \tag{A2.3}$$

where

$$\cosh\frac{\Omega_j}{2} = \cosh\frac{\eta_j}{2}\cosh\frac{\eta_{j-1}}{2}\cos\frac{\Delta\varphi_j+\Delta\psi_j}{2} - \sinh\frac{\eta_j}{2}\sinh\frac{\eta_{j-1}}{2}\cos\frac{\Delta\varphi_j-\Delta\psi_j}{2}.$$

Substituting back into (3.26) and taking $\varphi' = \psi' = 0$, the following expression is obtained:

$$\hat{K}(\eta'',\varphi'',\psi'',\eta',0,0;\sigma) = \frac{(\sinh\eta''\sinh\eta')^{1/2}}{8a}\exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{3\hbar^2}{32\mu a^2}\right)\sigma\right] \times \int_0^{2\pi}\int_{-2\pi}^{2\pi} U(\eta'',\varphi'',\psi'',\eta',0,0;\sigma)\exp(\mathrm{i}m\varphi'')\,\mathrm{d}\varphi''\,\mathrm{d}\psi'' \tag{A2.4}$$

where

$$U(\eta'',\varphi'',\psi'',\eta',0,0;\sigma) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left(\frac{i}{\hbar}S_j^g\right)\left(\frac{4\mu a^2}{2\pi\mathrm{i}\hbar\sigma_j}\right)^{1/2}\left(\frac{4\mathrm{i}\mu a^2}{2\pi\hbar\sigma_j}\right) \times \prod_{j=1}^{N-1} 2\pi^2(\sinh\eta_j\,\mathrm{d}\eta_j\,\mathrm{d}\varphi_j\,\mathrm{d}\psi_j/16\pi^2) \tag{A2.5}$$

$0 \leqslant \varphi < 2\pi,\ 0 \leqslant \eta < \infty,\ 0 \leqslant \psi < 4\pi$ and

$$S_j^g = \frac{4\mu a^2}{\sigma_j}\left(1-\cosh\frac{\Omega_j}{2}\right). \tag{A2.6}$$

The above result is just the path integral over the SU(1,1) group manifold isomorphic to a four-dimensional hyperboloid specified in terms of Eulerian angles. In this case, (A2.6) can be written as

$$S_j^g = \frac{4\mu a^2}{\sigma_j}(1 - \tfrac{1}{2}\,\mathrm{Tr}(\hat{g}_j)) = \frac{4\mu a^2}{\sigma_j}[1 - (\hat{e}_j, \hat{e}_{j-1})] \tag{A2.7}$$

where $\hat{g}_j = g_{j-1}^{-1}g_j$ for SU(1,1) group elements in the spinor representation and $(\hat{e}_j, \hat{e}_{j-1})$ is the scalar product of unit vectors on the hyperboloid. Also, the associate invariant measure is given by $\mathrm{d}g_j = (\sinh\eta_j\,\mathrm{d}\eta_j\,\mathrm{d}\varphi_j\,\mathrm{d}\psi_j/16\pi^2)$.

The subsequent evaluation of the path integral involves the expansion of the short-time propagator in matrix elements of UIRs of SU(1,1) given by the Bargmann Functions $d_{ab}^{l,\lambda}(\eta)$.

Appropriate to the problem considered in this paper is the fundamental continuous series with

$$l = -\tfrac{1}{2} + \mathrm{i}\rho \begin{cases} \rho \geqslant 0 & m = 0, \pm 1, \pm 2, \ldots : \lambda = 0 \\ \rho > 0 & m = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \ldots : \lambda = \tfrac{1}{2} \end{cases} \tag{A2.8}$$

where, furthermore, the series with $\lambda = 0$ is taken since the azimuthal quantum number $m = 0, \pm 1, \pm 2, \ldots$. In addition, $a = (m+m)/2 = m$ and $b = (m-m)/2 = 0$. Explicitly, the Bargmann functions [14] are given in terms of the hypergeometric functions as

$$\begin{aligned} d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta) &= \frac{1}{m!}\left(\frac{\Gamma(m+\frac{1}{2}+\mathrm{i}\rho)\Gamma(m+\frac{1}{2}-\mathrm{i}\rho)}{\Gamma(\frac{1}{2}+\mathrm{i}\rho)\Gamma(\frac{1}{2}-\mathrm{i}\rho)}\right)^{1/2} \tanh^m\left(\frac{\eta}{2}\right) \\ &\quad \times {}_2F_1\left[\frac{1}{2}+\mathrm{i}\rho, \frac{1}{2}-\mathrm{i}\rho; 1+m; -\sinh^2\left(\frac{\eta}{2}\right)\right] \end{aligned} \tag{A2.9}$$

and satisfy the orthogonality relation

$$\int_{\mathrm{SU}(1,1)} d_{m'0}^{-(1/2)+\mathrm{i}\rho',0}(g) d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(g)\,\mathrm{d}g = \frac{\delta(\rho-\rho')\delta_{mm'}}{2\rho\tanh(\pi\rho)}. \tag{A2.10}$$

The SU(1,1) propagator is then given by

$$\begin{aligned} \hat{K}(\eta'', \varphi'', \psi''; \eta', 0, 0; \sigma) &= \frac{(\sinh\eta''\sinh\eta')^{1/2}}{2a} \exp\left[-\frac{\mathrm{i}}{\hbar}\left(\frac{\rho^2\hbar^2}{2\mu a^2} - \frac{3\hbar^2}{32\mu a^2}\right)\sigma\right] \\ &\quad \times \int_0^\infty \mathrm{d}\rho\, 2\rho\tanh(\pi\rho) d_{m0}^{-(1/2)+\mathrm{i}\rho,0}(\eta'') d_{m0}^{-(1/2)+\mathrm{i}\rho,0*}(\eta') \end{aligned} \tag{A2.11}$$

which leads to (3.28).

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